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Introduction to Reaction-Diffusion Equations

Theory and Applications to Spatial Ecology and Evolutionary Biology

September 15, 2022

Springer Nature

Contents

Part I Linear Theory

Chapter 1
The maximum principle and principal eigenvalue for single equations

Abstract In this chapter we will discuss the maximum principle for single parabolic and elliptic equations, where we introduce the useful concept of super- and sub-solutions in a generalized sense which enables the gluing of families super- or subsolutions. Next, we derive the existence of principal eigenvalue by applying the Krein-Rutman Theorem. We will also determine the limit of principal eigenvalue as the diffusion rate tends to zero or infinity. Finally, we will give a characterization of the maximum principle based on the existence of a strict positive supersolution.

1.1 The maximum principle for single parabolic equations

Let Ω be a bounded domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, and let $\mathbf{n} = (n_i)_{i=1}^N$ be the unit outward normal vector on $\partial\Omega$. For $T > 0$, denote

$$\Omega_T = \Omega \times (0,T], \quad \overline{\Omega}_T = \overline{\Omega} \times [0,T], \quad S\Omega_T = \partial\Omega \times (0,T], \quad P\Omega_T = \overline{\Omega}_T \setminus \Omega_T.$$

Consider the linear parabolic operator in non-divergence form with continuous coefficients

$$u_t + \mathcal{L}u \equiv u_t - a^{ij}D_{ij}u - b^iD_iu - cu = f \quad \text{in } \Omega_T, \tag{1.1}$$

endowed with the oblique boundary condition

$$\mathcal{B}u \equiv p^iD_iu + p^0u = g \quad \text{on } S\Omega_T, \tag{1.2}$$

where we adopt the convention to sum over repeated indices. For each i, p^i satisfies

$$p^i(x,t)n_i(x) > 0 \quad \text{and} \quad p^0 \geq 0 \quad \text{on } \partial\Omega \times [0,T].$$

Setting $(p_i)_{i=1}^N$ to be the outward unit normal vector $\mathbf{n}$ on $\partial\Omega$, (1.2) reduces to

$$\mathbf{n} \cdot \nabla u + p^0u = g \quad \text{on } S\Omega_T,$$

and we obtain the Neumann boundary condition when $p^0 = 0$, or the Robin boundary condition when $p^0 \geq 0$ is nontrivial. We note that most of the results in this chapter continue to hold for the Dirichlet boundary condition.

In this chapter, we assume $a^{ij}, b^i, c \in C^0(\overline{\Omega}_T)$ and the uniform ellipticity condition; i.e. there exists some constant $\lambda_0 > 0$ such that

$$\lambda_0|\xi|^2 \leq a^{ij}(x,t)\xi_j\xi_j \quad \text{for all } \xi \in \mathbb{R}^N, \quad (x,t) \in \Omega_T. \tag{1.3}$$

We define the notions of classical and generalized super- and subsolutions for the initial-boundary-value problem

$$u_t + \mathcal{L}u = f(x,t,u,Du) \quad \text{in } \Omega_T, \quad \text{and} \quad \mathcal{B}u = g(x,t) \quad \text{on } S\Omega_T. \tag{1.4}$$

Definition 1.1.1 1. We say that $u \in C^{2,1}(\Omega_T)$ satisfies

$$u_t + \mathcal{L}u \leq f(x,t,u,Du) \quad \text{in } \Omega_T \tag{1.5}$$

in the classical sense if the inequality is satisfied everywhere in Ω_T. In this case, we call u a classical subsolution of $u_t + \mathcal{L}u = f$ in Ω_T. We similarly define the notion of classical supersolution by reversing the inequality (1.5).

2. We say that $u \in C^{1,0}(\overline{\Omega}_T)$ satisfies

$$\mathcal{B}u(x,t) \leq g(x,t) \quad (\text{resp. } \mathcal{B}u \geq g(x,t)) \quad \text{on } S\Omega_T,$$

in the classical sense if the inequality is satisfied everywhere on $S\Omega_T$.

3. We say that $u \in C(\overline{\Omega}_T)$ satisfies (1.5) in the generalized sense if, for each $(x_0,t_0) \in \Omega_T$, there exist a neighborhood U of (x_0,t_0) in Ω_T, and $\tilde{u} \in C^{2,1}(\overline{U})$ such that $\tilde{u} \leq u$ in U, $\tilde{u}(x_0,t_0) = u(x_0,t_0)$ and

$$\tilde{u}_t(x_0,t_0) + \mathcal{L}\tilde{u}(x_0,t_0) \leq f(x_0,t_0,\tilde{u}(x_0,t_0),D\tilde{u}(x_0,t_0)). \tag{1.6}$$

In such a case, we say that u is a generalized subsolution of $u_t + \mathcal{L}u = f$ in Ω_T.

4. We say that $u \in C(\overline{\Omega}_T)$ satisfies

$$u_t + \mathcal{L}u \geq f(x,t,u,Du) \quad \text{in } \Omega_T$$

in the generalized sense if, for each $(x_0,t_0) \in \Omega_T$, there exist a neighborhood U of (x_0,t_0) in Ω_T, and $\tilde{u} \in C^{2,1}(\overline{U})$ such that $\tilde{u} \geq u$ in U, $\tilde{u}(x_0,t_0) = u(x_0,t_0)$ and

$$\tilde{u}_t(x_0,t_0) + \mathcal{L}\tilde{u}(x_0,t_0) \geq f(x_0,t_0,\tilde{u}(x_0,t_0),D\tilde{u}(x_0,t_0)). \tag{1.7}$$

In such a case, we say that u is a generalized supersolution of $u_t + \mathcal{L}u = f$ in Ω_T.

5. We say that $u \in C(\overline{\Omega}_T)$ satisfies

$$u_t + \mathcal{L}u \leq f(x,t,u,Du) \quad \text{in } \Omega_T \quad \text{and} \quad \mathcal{B}u \leq g(x,t) \quad \text{on } S\Omega_T \tag{1.8}$$

in the generalized sense if for each $(x_0,t_0) \in \overline{\Omega}\times(0,T]$, there exist a neighborhood U of (x_0,t_0) in $\overline{\Omega}_T$ and a function $\tilde{u} \in C^{2,1}(\overline{U})$ such that $\tilde{u} \leq u$ in U, $\tilde{u}(x_0,t_0) =$

$u(x_0,t_0)$, and $\tilde{u}$ satisfies (1.6) in case $(x_0,t_0) \in \Omega_T$, or

$$\mathcal{B}\tilde{u}(x_0,t_0) \leq g(x_0,t_0) \quad \text{in case } (x_0,t_0) \in S\Omega_T.$$

In such a case, we say that u is a generalized subsolution of (1.4).

6. We say that $u \in C(\overline{\Omega}_T)$ satisfies

$$u_t + \mathcal{L}u \geq f(x,t,u,Du) \quad \text{in } \Omega_T \quad \text{and} \quad \mathcal{B}u \geq g(x,t) \quad \text{on } S\Omega_T \tag{1.9}$$

in the generalized sense if for each $(x_0,t_0) \in \overline{\Omega}\times(0,T]$, there exist a neighborhood U of (x_0,t_0) in $\overline{\Omega}_T$ and a function $\tilde{u} \in C^{2,1}(\overline{\Omega}_T)$ such that $\tilde{u} \geq u$ in U, $\tilde{u}(x_0,t_0) = u(x_0,t_0)$, and $\tilde{u}$ satisfies (1.7) in case $(x_0,t_0) \in \Omega_T$, or

$$\mathcal{B}\tilde{u}(x_0,t_0) \geq g(x_0,t_0) \quad \text{in case } (x_0,t_0) \in S\Omega_T.$$

In such a case, we say that u is a generalized supersolution of (1.4).

Remark 1.1.2 1. We define $u_t+\mathcal{L}u < f(x,t,u,Du)$ in Ω_T in the generalized sense if statement 3 in the above definition hold with the inequality in (1.6) being replaced by a strict inequality. We also define $u_t + \mathcal{L}u > f(x,t,u,Du)$ analogously.

2. If $u_t+\mathcal{L}u \leq f(x,t,u,Du)$ in Ω_T (resp. $u_t+\mathcal{L}u < f(x,t,u,Du)$ in Ω_T or $\mathcal{B}u \leq 0$ in $S\Omega_T$) in the classical sense, then it automatically holds in the generalized sense.
3. Let u_i $(i = 1,2)$ be two classical subsolutions of $u_t + \mathcal{L}u = f(x,t,u,Du)$ in Ω_T, then $u = \max\{u_1,u_2\}$ is a generalized subsolution of the same equation in Ω_T.
4. Let u_i $(i = 1,2)$ be two classical supersolutions of $u_t + \mathcal{L}u = f(x,t,u,Du)$ in Ω_T, then $u = \min\{u_1,u_2\}$ is a generalized supersolution of the same equation in Ω_T.
5. Let A_1, A_2 be relatively open subsets in $\overline{\Omega}_T$ such that $\overline{\Omega}_T \subset (A_1 \cup A_2)$. Suppose

   a. u_i $(i = 1,2)$ are generalized subsolutions of $u_t+\mathcal{L}u = f(x,t,u,Du)$ in $A_i\cap\Omega_T$, and
   b. $u_1 < u_2$ on $(\partial A_1) \cap A_2 \cap \Omega_T$ and $u_1 > u_2$ on $A_1 \cap (\partial A_2) \cap \Omega_T$,

   then $u = \max\{u_1,u_2\}$ belongs to $C(\Omega_T)$ and is a generalized subsolution of the same equation in Ω_T. A similar statement holds if u_i are subsolutions of $u_t + \mathcal{L}u = f(x,t,u,Du)$ in $A_i \cap \Omega_T$ and $\mathcal{B} = g$ on $A_i \cap S\Omega_T$. A similar statement holds for generalized supersolutions.
6. For operators in divergence form with measurable coefficients, the notion of weak super- and subsolutions can be defined by integration by parts with nonnegative test functions. See [2] and [10] (and also Problem **??**) for gluing of the weak super- and subsolutions solutions in the H^1 framework. For operators in non-divergence form with continuous coefficients, a general theory is developed for super- and subsolutions in the viscosity sense [9]. Here we simply observe that the classical notions of super- and subsolutions can be somewhat generalized (with no change to the proofs) without using more technical machinery. This covers many usual constructions [7, 25].

Theorem 1.1.3 *(Weak Maximum Principle) Let $u \in C(\overline{\Omega}_T)$ be given.*

1. *If* $u_t + \mathcal{L}u \le 0$ *holds in the generalized sense in* Ω_T *and* $u \le 0$ *in* $P\Omega_T$*, then*

$$u \le 0 \quad \text{in } \Omega_T,$$

2. *If* $u_t + \mathcal{L}u \le 0$ *holds in the generalized sense in* Ω_T *and* $c \le 0$ *in* Ω_T*, then*

$$\max_{\overline{\Omega}_T} u \le \max_{P\Omega_T} u^+,$$

where $u^+ := \max\{u, 0\}$.

Remark 1.1.4 Let $u \in C(\overline{\Omega}_T)$ be given. Suppose $u_t + \mathcal{L}u \le 0$ and $u_t + \mathcal{L}u \ge 0$ in Ω_T in the generalized sense, and that $c \le 0$. Then

$$\max_{\overline{\Omega}_T} |u| \le \max_{P\Omega_T} |u| \quad \text{in } \Omega_T.$$

See Problem **??**.

Remark 1.1.5 For the weak maximum principle when Ω is an unbounded domain, or the entire space, additional growth condition at infinity has to be imposed. See Section 6.2 in Chapter 6 for the counterpart of Theorem 1.1.3 in that setting.

Proof First, we prove the second assertion. It is enough to prove that

$$\max_{\overline{\Omega}_T}(u - \epsilon t) \le \max_{P\Omega_T}(u - \epsilon t)^+ \quad \text{for each } \epsilon > 0.$$

Suppose to the contrary that there exist $\epsilon > 0$ and $(x_0, t_0) \in \Omega_T$ such that $\max_{\overline{\Omega}_T}(u - \epsilon t) = u(x_0, t_0) - \epsilon t_0 > \max_{P\Omega_T}(u - \epsilon t)^+ \ge 0$. For the point (x_0, t_0), take the smooth function $\tilde{u}$ according to the definition of generalized subsolution, so that

$$\begin{cases} \tilde{u} \le u & \text{in a neighborhood of } (x_0, t_0), \\ \tilde{u}(x_0, t_0) = u(x_0, t_0) > 0 \quad \text{and} & \tilde{u}_t + \mathcal{L}\tilde{u} \le 0 \quad \text{at } (x_0, t_0). \end{cases} \tag{1.10}$$

Since $u(x, t) - \epsilon t$ has a local maximum at (x_0, t_0), it follows that $\tilde{u}(x, t) - \epsilon t$ also has a local maximum at (x_0, t_0). Hence,

$$\tilde{u}_t \ge \epsilon, \quad a^{ij} D_{ij}\tilde{u} + b^j D_j \tilde{u} \le 0, \quad c\tilde{u} \le 0, \quad \text{at } (x_0, t_0),$$

so that

$$\tilde{u}_t + \mathcal{L}\tilde{u} \ge \epsilon > 0 \quad \text{at } (x_0, t_0),$$

which is in contradiction with (1.10). This proves the second assertion.

For the first assertion, let $v(x, t) = e^{-t \sup c} u(x, t)$, then v satisfies

$$v_t - a^{ij} D_{ij} v - b^j D_j v - \tilde{c} v \le 0 \quad \text{in } \Omega_T$$

in the generalized sense, where $\tilde{c} = c - \sup_{\Omega_T} c \le 0$ in Ω_T. We can then repeat the previous step to deduce that

$$\max_{\overline{\Omega}_T} v(x,t) \le \max_{P\Omega_T} v(x,t) = \max_{P\Omega_T} e^{t\,\sup_{\Omega_T}} u(x,t) \le 0.$$

This is equivalent to $u(x,t) \le 0$ in Ω_T. This proves the first assertion. □

We also prove that, when $c < 0$ everywhere, then a generalized subsolution u cannot attain a positive local maximum in the interior.

Lemma 1.1.6 *Let Ω be a domain in $\mathbb{R}^N$ which is possibly unbounded. Suppose u satisfies $u_t + \mathcal{L}u \le 0$ in the generalized sense in Ω_T, where $a^{ij}, b^i, c \in C_{loc}(\Omega_T)$ such that $a^{ij}\xi_i\xi_j > 0$ for all $\xi = (\xi_i) \in \mathbb{R}^N$. If $c < 0$ in Ω_T (but not necessarily bounded from below), then u cannot attain a positive local maximum at an interior point.*

Proof Suppose to the contrary that u has a local maximum point $(x_0, t_0) \in \Omega_T$ such that $u(x_0, t_0) > 0$. Then by the definition of generalized subsolution, there exists a smooth function $\tilde{u}$ such that $\tilde{u} \le u$ in a neighborhood of (x_0, t_0) and that equality holds at (x_0, t_0). Hence, $\tilde{u}$ also attains a positive local maximum value at (x_0, t_0), so that

$$\tilde{u}_t \ge 0, \quad a^{ij} D_{ij}\tilde{u} + b^i D_i \tilde{u} \le 0, \quad c\tilde{u} < 0 \quad \text{at } (x_0, t_0).$$

This contradicts the fact that $\tilde{u}_t + \mathcal{L}u \le 0$. □

Having proved the weak maximum principle for generalized super- and subsolutions, we can derive the Boundary Point Lemma and Strong Maximum Principle, and Growth Lemma, since they are all predicated on being able to compare u with a suitable classical barrier function via the weak maximum principle.

Theorem 1.1.7 *(Boundary Point Lemma) Let $R > 0$ and $Y = (y, s) \in \mathbb{R}^{N+1}$ be given. Consider the lower paraboloid*

$$P_R = P_R(Y) = \{X = (x,t) : |x-y|^2 + (s-t) < R^2, \text{ and } t < s\}.$$

Suppose $u \in C(\overline{P}_R)$ satisfies $u_t + \mathcal{L}u \ge 0$ in P_R in the generalized sense, and that there is $X_1 = (x_1, s)$ with $|x_1 - y| = R$ such that

$$u(X_1) > u(X) \quad \text{and} \quad c(X)u(X_1) \le 0 \quad \text{for all } X \in P_R. \tag{1.11}$$

Then $\beta \cdot \nabla_x u(X_1) > 0$ for any vector β such that $\beta \cdot (x_1 - y) > 0$, in the sense

$$\liminf_{h \to 0+} \frac{u(x_1, s) - u(x_1 - \beta h, s)}{h} > 0.$$

Proof This is originally due to Nirenberg [30], who used ellipsoid instead of paraboloid. See [26, Chap. II, Lemma 2.8] for the proof. Here we observe that the proof works not only for classical subsolutions, but also generalized subsolutions. This is based on observing that $u(X_1) - u$ and the auxiliary function $h_R(r) = e^{-\alpha r^2} - e^{-\alpha R^2}$ form a pair of super- and sub-solution in the domain $P_R \backslash \overline{P_{R/2}}$, in the generalized sense. Here

$$\begin{cases} X = (x_1, ..., x_N, t) \quad \text{and} \quad X_1 = (x_{1,1}, ..., x_{N,1}, t_1), \\ r^2 = |t - t_1| + \sum_{i=1}^N |x_i - x_{i,1}|^2. \end{cases}$$

Therefore, fixing $\epsilon > 0$ so that

$$u(x_1, s) - u(x, t) \geq \epsilon h_R(r) \quad \text{on } \{X \in \partial P_{R/2},\ t < s\}$$

and recalling that $u(x_1, s) - u(x, t) \geq 0 = \epsilon h_R(r)$ on $\{X \in \partial P_R,\ t < s\}$, one may apply the weak maximum principle to obtain

$$u(x_1, s) - u(x, s) \geq \epsilon h_R(r)$$

for all $(x, s) \in P_R \setminus P_{R/2}$. □

Lemma 1.1.8 *(Growth Lemma due to Krylov-Safonov) Let $R > 0$ and $\alpha > 0$ be positive constants and set*

$$Q = \{(x, t) : |x| < R,\ -\alpha R^2 < t < 0\}.$$

Then there exists a uniform positive constant κ such that for each

$$u \in \left\{\tilde{u} \in C(\overline{\Omega}_T) : \ \tilde{u} \geq 0 \ \text{ in } Q, \ \text{ and } \tilde{u}_t + \mathcal{L}\tilde{u} \leq 0 \ \text{ in } \ Q \ \text{ in the generalized sense}\right\}$$

if

$$u \geq h \quad \text{in } \{|x| < \epsilon R,\ t = -\alpha R^2\}, \quad \text{for some } h > 0,\ 0 < \epsilon < 1,$$

then

$$u \geq \frac{\epsilon^\kappa h}{2} \quad \text{in } \{|x| < R/2,\ t = 0\}.$$

Proof See [26, Lemma 2.6]. □

Theorem 1.1.9 *(Strong Maximum Principle) Assume*

$$u_t + \mathcal{L}u \leq 0 \quad \text{in } \Omega_T, \quad \mathcal{B}u \leq 0 \quad \text{on } S\Omega_T, \quad \text{in the generalized sense.}$$

1. If $u(x, 0) \leq 0$ in Ω, then $u \leq 0$ in Ω_T.
2. If, in addition, $u(x_0, t_0) = 0$ for some $(x_0, t_0) \in \overline{\Omega} \times (0, T]$, then

$$u \equiv 0 \quad \text{in } \overline{\Omega} \times (0, t_0].$$

Proof We first show part 1. By replacing $u(x, t)$ with $e^{-t \sup c} u(x, t)$ if necessary, we may assume $c \leq 0$ in Ω_T. Suppose $u(x, 0) \leq 0$ in Ω, we need to show that $u \leq 0$ in Ω_T. It suffices to show that $v(x, t) = u(x, t) - \epsilon t$ is negative in Ω_T for each $\epsilon > 0$. Suppose to the contrary that there exists $\epsilon > 0$ such that $\sup_{\Omega_T} v = v(x_0, t_0) > 0$ for some $(x_0, t_0) \in \overline{\Omega}_T$, then necessarily $t_0 > 0$.

Step 1. We first discuss the case $x_0 \in \operatorname{Int}\Omega$. In this case, there exist a neighborhood U of (x_0, t_0) and a smooth function $\tilde{u}$ satisfying

$$\tilde{u}_t + \mathcal{L}\tilde{u} \le 0 \quad \text{at } (x_0, t_0) \tag{1.12}$$

and that

$$v(x,t) = u(x,t) - \epsilon t \ge \tilde{u}(x,t) - \epsilon t, \quad \text{such that equality holds at } (x_0, t_0).$$

Since v attains a positive local maximum point at (x_0, t_0), the smooth function $\tilde{u}(x,t) - \epsilon t$ also attains a positive maximum value at (x_0, t_0), i.e.

$$\tilde{u}_t \ge \epsilon, \quad a^{ij} D_{ij}\tilde{u} + b^j D_j \tilde{u} \le 0, \quad c\tilde{u} \le 0, \quad \text{at } (x_0, t_0),$$

so that

$$\tilde{u}_t + \mathcal{L}\tilde{u} \ge \epsilon > 0 \quad \text{at } (x_0, t_0),$$

which is in contradiction with (1.12).
Step 2. Suppose $\sup_{\Omega_T} v > 0$ and $v < \sup_{\Omega_T} v$ in Ω_T, i.e. v attains its positive maximum at some $(x_0, t_0) \in \partial\Omega \times (0, T]$. By applying the Boundary Point Lemma (Theorem 1.1.7) we deduce that

$$p^j D_j \tilde{u} \ge p^j D_j v > 0 \quad \text{at } (x_0, t_0).$$

But this is impossible since, according to the definition of generalized subsolutions,

$$p^0 \tilde{u} \ge 0 \quad \text{and} \quad p^j D_j \tilde{u} + p^0 \tilde{u} \le 0 \quad \text{at } (x_0, t_0).$$

Step 3. By Steps 1 and 2, $v(x,t) = u(x,t) - \epsilon t \le 0$ in Ω_T for all $\epsilon > 0$. Letting $\epsilon \to 0$, we conclude that $u \le 0$ in Ω_T. In summary, $u(x, 0) \le 0$ in Ω implies $u \le 0$ in Ω_T. This proves assertion 1.
Step 4. Suppose that $u(x_0, t_0) = 0$ for some $(x_0, t_0) \in \Omega \times (0, T]$. Then the Growth Lemma implies $u(x,t) \equiv 0$ in $\Omega \times [0, t_0)$. By continuity, we have $u(x,t) \equiv 0$ in $\overline{\Omega} \times [0, t_0]$.
Step 5. Suppose that $u(x_0, t_0) = 0$ for some $(x_0, t_0) \in S\Omega_T$ and $u < 0$ in $\Omega \times [0, t_0]$. We may repeat the arguments in Step 2 to obtain a contradiction. This completes the proof. □

1.2 The comparison principle for semilinear equations

Consider the semilinear parabolic equation:

$$u_t + \mathcal{L}u = f(x, t, u, Du) \quad \text{in } \Omega_T, \quad \mathcal{B}u = 0 \quad \text{on } S\Omega_T, \tag{1.13}$$

where $\mathcal{L}$ is a non-divergence form operator with continuous coefficients, $f(x,t,s,p)$ is C^α in x, $C^{\alpha/2}$ in t with $\alpha \in (0,1)$, and C^1 in (s,p); $\mathcal{B}u$ is the oblique boundary operator.

Corollary 1.2.1 *Suppose $u, v \in C(\overline{\Omega}_T)$ satisfies*

$$\begin{cases} u_t + \mathcal{L}u \le f(x,t,u,Du) \quad \text{and} \quad v_t + Lv \ge f(x,t,v,Dv) & \text{in } \Omega_T, \\ \mathcal{B}(u-v) \le 0 & \text{in } S\Omega_T \end{cases} \tag{1.14}$$

in the generalized sense. If $u(x,0) \le v(x,0)$ in Ω, then $u \le v$ in Ω_T.

Proof Suppose $u, v \in C^{2,1}(\overline{\Omega}_T)$ and satisfies (1.14) in the classical sense. It suffices to observe that $w = u - v$ satisfies

$$w_t + \mathcal{L}w \le b(x,t)\cdot Dw + c(x,t)w \quad \text{in } \Omega_T, \quad \mathcal{B}w \le 0 \quad \text{on } S\Omega_T$$

in the generalized sense, where

$$\begin{cases} b(x,t) = \int_0^1 D_4 f(x,t,u(x,t), \xi Du(x,t) + (1-\xi)Dv(x,t))\, d\xi, \\ c(x,t) = \int_0^1 D_3 f(x,t,\xi u(x,t) + (1-\xi)v(x,t), Dv(x,t))\, d\xi, \end{cases}$$

and $D_i f$ is the partial derivative of f with respect to the i-th variable. The conclusion follows from part 1 of Theorem 1.1.9. We leave the general case as an exercise (see Problem **??**). □

Definition 1.2.2 Suppose f is independent of t, i.e. $f = f(x,u,Du)$. We say that $\underline{\theta} \in C^2(\Omega) \cap C^1(\overline{\Omega})$ is a subequilibrium of (1.13) in the classical sense if it satisfies in the classical sense the following:

$$\mathcal{L}\underline{\theta} \le f(x,\underline{\theta},D\underline{\theta}) \quad \text{in } \Omega, \qquad \mathcal{B}\underline{\theta} \le 0 \quad \text{on } \partial\Omega. \tag{1.15}$$

And we say that $\underline{\theta} \in C(\overline{\Omega})$ is a subequilibrium of (1.13) in the generalized sense if for each $x_0 \in \overline{\Omega}$, there exist a neighborhood U of x_0 in $\overline{\Omega}$ and a function $\tilde{\theta} \in C^{2,1}(\overline{\Omega})$ such that $\underline{\theta} \ge \tilde{\theta}$ in U, $\underline{\theta}(x_0) = \tilde{\theta}(x_0)$, and $\tilde{\theta}$ satisfies

$$\mathcal{L}\tilde{\theta}(x_0) \le f(x_0, \tilde{\theta}(x_0), D\tilde{\theta}(x_0)) \quad \text{in case } x_0 \in \Omega,$$

or

$$\mathcal{B}\tilde{\theta}(x_0) \le g(x_0) \quad \text{in case } x_0 \in \partial\Omega.$$

Analogously, we define the notion of superequilibrium in the classical and generalized sense by reversing the above inequalities.

Remark 1.2.3 Suppose f is independent of t. If $\underline{\theta}$ is a subequilibrium of (1.13) in the classical sense (resp. generalized sense), then it is automatically a subsolution of (1.13) in the classical sense (resp. generalized sense). A similar statement holds for superequilibrium.

Corollary 1.2.4 *(Monotone iteration method due to D. Sattinger [32])*

Assume that $f = f(x, u, p)$ is independent of t. Suppose $u \in C^{2,1}(\Omega_T) \cap C^{1,0}(\overline{\Omega} \times (0, T]) \cap C(\overline{\Omega_T})$ is a classical solution of $u_t + \mathcal{L}u = f(x, u, Du)$ and $u_0(x)$ is a subequilibrium (resp. superequilibrium) in the generalized sense, then $t \mapsto u(x, t)$ is nondecreasing (resp. nonincreasing) for every $x \in \Omega$. Moreover, if $T = \infty$, and

$$\sup_{t\geq 0} \|u(\cdot, t)\|_{L^\infty(\Omega)} < +\infty, \quad \limsup_{t\to\infty} \|f(\cdot, u(\cdot, t), Du(\cdot, t))\|_{L^\infty(\Omega)} < +\infty,$$

then $u_\infty(x) := \lim_{t\to\infty} u(x, t)$ converges in $C(\overline{\Omega})$, such that $u_\infty \in W^{2,p}(\Omega)$ satisfies $\mathcal{B}u_\infty = 0$ in the classical sense, and is a strong solution of

$$\mathcal{L}u_\infty = f(x, u_\infty, Du_\infty) \quad \text{in } \Omega. \tag{1.16}$$

Proof By Corollary 1.2.1, we have

$$u(x, t_0) \geq u_0(x) \quad \text{for any } (x, t_0) \in \Omega \times (0, T).$$

Next, fix $t_0 \in (0, T)$ and apply the comparison principle to the solutions to $u_t + \mathcal{L}u = f(x, u)$ with initial conditions $u_0(x)$ and $u(x, t_0)$, we deduce that

$$u(x, t + t_0) \geq u(x, t) \quad \text{for any } x \in \Omega,\ t \in (0, T - t_0].$$

This proves the first part.

For the second part, observe first that since $u(x, t)$ is monotone and bounded as $t \to \infty$, the limit $u_\infty(x) := \lim_{t\to\infty} u(x, t)$ exists in the pointwise sense and belongs to $L^\infty(\Omega)$. Next, by the boundedness of u and $f(x, t, u, Du)$ in $L^\infty(\Omega \times [0, \infty))$, we can apply the L^p estimates to deduce that

$$\sup_{t\geq 1} \|u\|_{W^{2,1,p}(\Omega\times[t,t+1])} < \infty.$$

This means $u(x, t + k) - u_\infty(x) \to 0$ as $k \to \infty$ weakly in $W^{2,1,p}(\Omega \times [0, 1])$ and (thanks to Sobolev embedding) strongly in $C^{1+\alpha,(1+\alpha)/2}(\overline{\Omega} \times [0, 1])$. It is standard to check that u_∞ is a strong solution of (1.16), and classical solution of $\mathcal{B}u_\infty = 0$ on $\partial\Omega$. ◻

Theorem 1.2.5 *(Interior Harnack inequality) Let $u \in W^{2,1}_{N+1,loc}(\Omega \times (0, T)) \cap C(\overline{\Omega_T})$ be a strong solution of $u_t + \mathcal{L}u = 0$ in $\Omega \times (0, T)$ on Ω_T. Suppose u is nonnegative on Ω_T. For each compact set $K \subset \Omega$ and $\delta > 0$, there exists $C_1 = C_1(K)$ such that*

$$\sup_K u(\cdot, s) \leq C_1 \inf_K u(\cdot, t) \quad \text{for every } s, t \in [0, T] \text{ satisfying } s \geq \delta,\ t - s \geq \delta. \tag{1.17}$$

If further assume that $u \in C^{1,0}(\overline{\Omega_T})$ and $\mathcal{B}u = 0$ on $S\Omega_T$ in the classical sense, then C_1 can be chosen independent of K, i.e.

$$\sup_{\Omega} u(\cdot, s) \le C_1 \inf_{\Omega} u(\cdot, t) \quad \textit{for every } s, t \in [0, T] \textit{ satisfying } s \ge \delta,\ t - s \ge \delta. \tag{1.18}$$

Proof For the proof of (1.17) see, e.g. [12, Theorem 2.2]. The estimate (1.18) is contained in the proof of [21, Theorem 2.5] (which is, in turn, based on the weak Harnack inequalities [21, Lemma 3.5] and [27, Theorem 7.10]). Here, we shall give a quick proof of (1.18) for the special case of u being a nonnegative solution of the Neumann problem

$$\begin{cases} u_t - \Delta u = c(x,t)u & \text{in } \Omega_T, \\ \partial_n u := n_i(x) D_i u = 0 & \text{on } S\Omega_T, \\ u(x,0) = u_0(x) & \text{in } \Omega, \end{cases} \tag{1.19}$$

where Δ is the Laplace operator; Ω is a bounded domain with smooth boundary $\partial\Omega$; $n(x) = (n_i(x))$ is the outward unit normal vector on $\partial\Omega$; and $c \in L^\infty(\Omega_T)$. In this case, let $\tilde{u}$ be an extension of u obtained via reflecting across the spatial boundary $S\Omega_T$. It is a standard fact that $\tilde{u}$ satisfies a uniformly parabolic equation in $\tilde{\Omega} \times [0, T]$, for some bounded open set $\tilde{\Omega}$ containing the closure of Ω. Now, we may deduce (1.18) from (1.17). □

Next, we give a useful result due to J. Húska [21, Theorem 2.5], for nonnegative strong solutions. Note that classical solutions are automatically strong solutions.

Theorem 1.2.6 *(Harnack principle) Let $c \in L^\infty(\Omega_T)$, $T \ge \delta_0$. Then there exists C (depending on δ_0 among other things, but independent of T) such that for every nonnegative strong solution $u \in W^{2,1}_{N+1,loc}(\Omega \times (0,T)) \cap C^{0,1}(\overline{\Omega}_T)$ of $u_t + \mathcal{L}u = 0$ in $\Omega \times (0, T)$, which satisfies $\mathcal{B}u = 0$ on $S\Omega_T$ in the classical sense, we have*

$$\sup_{x \in \Omega} u(x,t) \le C \inf_{x \in \Omega} u(x,t) \quad \textit{for each } t \in [\delta_0, T].$$

Proof By comparing the solution u with the supersolution $\overline{u} = e^{\|c\|_\infty (t - t_1)}$ over the set $\Omega \times [t_1, t_2]$, we deduce

$$\sup_{x \in \Omega} u(x, t_2) \le e^{\|c\|_\infty (t_2 - t_1)} \sup_{x \in \Omega} u(x, t_1) \quad \text{for every } 0 \le t_1 < t_2 \le T. \tag{1.20}$$

Next, we apply (1.18) of Theorem 1.2.5 and obtain

$$\sup_{\Omega} u(\cdot, t - \delta_0) \le C \inf_{\Omega} u(\cdot, t). \tag{1.21}$$

Finally, we combine (1.20) and (1.21) to deduce that there exists $C' = C e^{\|c\|_\infty \delta_0}$ such that

$$\sup_{\Omega} u(\cdot, t) \le e^{\|c\|_\infty \delta_0} \sup_{\Omega} u(\cdot, t - \delta_0) \le C' \inf_{\Omega} u(\cdot, t). \tag{1.22}$$

This completes the proof. □

Fix $X_0 = (x_0, t_0)$. Define

$$Q(X_0, R) = \{(x,t) : |x - x_0| < R, t_0 - R^2 < t < t_0\}.$$

Theorem 1.2.7 *(Local maximum principle) If $u \in W^{2,1}_{N+1}(Q(X_0, R))$ satisfies $u_t + \mathcal{L}u \le f$ in $Q(X_0, R)$ in the strong sense, then for any $p > 0$ and $r \in (0,1)$, there is a constant C determined only by p, r and bounds of the coefficients such that*

$$\sup_{Q(X_0, rR)} u \le C \left[\left(R^{-N-2} \int_{Q(X_0,R)} (u^+)^p \, dX \right)^{1/p} + R^{N/(N+1)} \|f\|_{L^{N+1}(Q(X_0,R))} \right].$$

Proof See [26, Theorem 7.36]. □

Corollary 1.2.8 *Suppose $u \in W^{2,1}_{N+1}(\Omega_T)$ satisfies $u_t - \Delta u + b^i D_i u + cu \le f$ in Ω_T in the strong sense, and satisfies the Neumann boundary condition $n_i D_i u = 0$ on $S\Omega_T$ in the classical sense. Then for any $p > 0$ and $t_0 \in (0, T)$, there is a constant C determined by p, the domain Ω_T, and bounds of the coefficients such that*

$$\sup_{\Omega \times (t_0, T]} u \le C \left[\|u^+\|_{L^p(\Omega_T)} + \|f\|_{L^{N+1}(\Omega_T)} \right].$$

Proof This follows by first reflecting across the spatial boundary, to extend u such that it is a strong solution to $u_t + \Delta u \le f$ in a larger cylinder domain Ω'_T, and that

$$\sup_{\Omega'_T} u^+ \le C \sup_{\Omega_T} u^+.$$

The conclusion follows from the local maximum principle (Theorem 1.2.7), and we omit the details. □

1.3 The principal eigenvalue for linear elliptic operators

We recall the classical Krein-Rutman Theorem for positive compact linear operators.

Definition 1.3.1 Let K be a subset of a Banach space X.

1. The set K is a *cone* if (i) K is closed and convex , (ii) $\mu K \subset K$ for all $\mu \ge 0$, and (iii) $K \cap (-K) = \{0\}$.
2. K is a *total cone* if it is a cone and $K - K = \{x - y : x, y \in K\}$ is dense in X.
3. K is a *solid cone* if it is a cone with nonempty interior.

Definition 1.3.2 The cone K in X induces a partial ordering of X. For $x, y \in X$, we write

$$x \le y \qquad \text{(resp.} \qquad x < y, \qquad x \ll y)$$

if

$$y - x \in K \quad (\text{resp. } y - x \in K \setminus \{0\}, \quad y - x \in \operatorname{Int} K).$$

In such a case, we say that X is an ordered Banach space with order induced by the cone K.

We state the classical theorems due to Krein and Rutman [24]. The proofs can be found in Appendix B.

Theorem 1.3.3 (Krein and Rutman) *Suppose that X is a real ordered Banach space with order induced by a total cone K, and $T : X \to X$ is a compact linear operator with spectral radius $r(T)$. If $T(K) \subset K$ and $r(T) > 0$, then there exist $x_0 \in K \setminus \{0\}$ and $f_0 \in K^* \setminus \{0\}$ such that*

$$Tx_0 = rx_0 \quad \textit{and} \quad T^* f_0 = r f_0,$$

where $r = r(T)$, T^* is the adjoint of T, and K^* is the adjoint cone

$$K^* = \{f \in X^* : f(x) \geq 0 \quad \textit{for all } x \in K\}.$$

Theorem 1.3.4 *Suppose that X is a real ordered Banach space with order induced by the cone K. If K is a solid cone and $T : X \to X$ is a compact linear operator which is strongly monotone, i.e. $T(K \setminus \{0\}) \subset \operatorname{Int} K$, then the following statements hold.*

1. *The spectral radius $r(T)$ is positive and is a simple eigenvalue with an eigenvector $v \in \operatorname{Int} K$. Moreover, there is no other eigenvalue with an eigenvector in $K \setminus \{0\}$.*
2. *There exists $\epsilon > 0$ such that $|\lambda| < r(T) - \epsilon$ for all eigenvalues $\lambda \in \mathbb{C} \setminus \{r(T)\}$.*

In the following section, we will choose to work with eigenfunctions in the strong sense. This is consistent with our applying Krein-Rutman Theorem to the inverse of the elliptic operator $\mathcal{L}$ in the space $C(\bar{\Omega})$ of continuous function on $\bar{\Omega}$. Precisely, the eigenfunctions for the elliptic problem thus belong to $\cap_{p>1} W^{2,p}(\Omega)$. We remark that if all the coefficients are assumed to be Hölder continuous, then the eigenfunctions will be classical by virtue of the Schauder theory.

We will need the following touching lemma for strong solutions to elliptic problems with oblique derivative conditions.

Lemma 1.3.5 *Let a^{ij}, b^i, c, p^i, p^0 be independent of time, and let $p > N$. Suppose $w \in W^{2,p}(\Omega)$ is a nonnegative, strong solution to*

$$\begin{cases} \mathcal{L}w \geq 0 & \textit{in } \Omega, \\ \mathcal{B}w \geq 0 & \textit{on } \partial\Omega. \end{cases}$$

If $\inf_\Omega w = 0$, then $w \equiv 0$ in $\overline{\Omega}$.

Note that by Sobolev imbedding, w satisfies $\mathcal{B}w \geq 0$ in the classical sense on $\partial\Omega$.

Proof By the interior Harnack inequality for nonnegative strong supersolutions [15, Theorem 9.22] asserts that, for each x_0 and $R > 0$ such that $B_{2R}(x_0) \subset \Omega$,

$$\left(\frac{1}{|B_R|}\int_{B_R} w^p\right)^{1/p} \leq C \inf_{B_R} w$$

where p and C are positive constants independent of w. It follows that the set $A = \{x \in \Omega : w(x) = 0\}$ is open in Ω. Since $w \in C(\bar{\Omega})$, we see that A is also closed. Hence $A = \Omega$ or A is empty. In the first case, $w \equiv 0$ and we are done. In the latter case $w > 0$ in Ω and $w(x_0) = 0$ for some $x_0 \in \partial\Omega$. By the Hopf boundary lemma[1], this implies that $\mathcal{B}w = p^i D_i w > 0$ at x_0 this is a contradiction. Hence it must hold that $w \equiv 0$. □

Theorem 1.3.6 *If a^{ij}, b^i, c, p^i, p^0 are independent of time, then the elliptic eigenvalue problem*

$$\begin{cases} \mathcal{L}\phi = \mu\phi & \text{in } \Omega, \\ \mathcal{B}\phi = 0 & \text{on } \partial\Omega \end{cases} \tag{1.23}$$

has a principal eigenvalue μ_1, in the sense that

1. *$\mu_1 \in \mathbb{R}$ is a simple eigenvalue of* (1.23)*,*
2. *$\mu_1 < \inf \operatorname{Re} \mu$, where the infimum is taken over all eigenvalues μ of* (1.23) *which is distinct from μ_1,*
3. *The eigenfunction $\phi_1(x)$ corresponding to μ_1 can be chosen such that $\phi_1 > 0$ in $\overline{\Omega}$,*
4. *If μ is an eigenvalue of* (2.1) *with a nonnegative eigenfunction, then $\mu = \mu_1$.*

Remark 1.3.7 It follows from Theorem 1.3.6 that if the boundary value problem

$$\mathcal{L}w = \mu_0 w \ \text{ in } \Omega, \quad \mathcal{B}w = 0 \ \text{ on } \partial\Omega$$

has a positive solution for some $\mu_0 \in \mathbb{R}$, then μ_0 is necessarily the principal eigenvalue of (1.23).

Remark 1.3.8 In fact, the same conclusion holds for the weighted eigenvalue problem:

$$\begin{cases} \mathcal{L}\phi = \mu\beta\phi & \text{in } \Omega, \\ \mathcal{B}\phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.24}$$

where $\beta \in C(\overline{\Omega})$ is strictly positive on $\overline{\Omega}$. See Problem **??**.

Proof In the following, we present a proof based on applying the Krein-Rutman theorem to the inverse of elliptic operator $\mathcal{L}$. A shorter proof is possible by considering

[1] By Alexandrov-Bakelman-Pucci maximum principle [15, Theorem 9.1], one can repeat the proof of [15, Lemma 3.4] or [28, Lemma 1.24] to compare the supersolution w (in the strong sense) with the barrier function.

the semigroup operator of the parabolic problem; this will be presented in Chapter 2. If the elliptic problem is not linear (e.g. $\mathcal{L}$ is only positively homogeneous of degree one and is well-defined only in a cone), then it may be necessary to use the proof based on the semigroup operator; see [18] for an example in this situation.

Recall that $\mathcal{L} = -a^{ij}D_{ij} - b^i D_i - c$, where $a^{ij}, b^i, c \in C^0(\overline{\Omega})$ satisfy the ellipticity condition (1.3). Replacing μ by $\mu - \|c\|_{C(\overline{\Omega})} - 1$ in (1.23), we may assume without loss of generality that $c \le -1$ in Ω.

Consider the inhomogeneous linear problem

$$\begin{cases} \mathcal{L}v = f & \text{in } \Omega, \\ \mathcal{B}v = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.25}$$

Given $f \in C(\overline{\Omega})$, it is well-known that (1.25) has a unique solution $v \in \cap_{p>1} W^{2,p}(\Omega)$ (see, e.g. [28, Theorem 6.30]). Hence, we can define the linear operator $T : C(\overline{\Omega}) \to C(\overline{\Omega})$ such that $Tf = v$. Moreover, for each $p > 1$, there exists C such that

$$\|v\|_{W^{2,p}(\Omega)} \le C\|f\|_{L^p(\Omega)} \le C'\|f\|_{C(\overline{\Omega})}.$$

The Sobolev embedding theorem (see [15, Chapter 7]) asserts that $W^{2,p}(\Omega)$ is compactly embedded in $C^1(\overline{\Omega})$ for $p > N$. We can therefore take $p > N$ in the above inequality and deduce that the linear operator T is compact.

Next, we remark that T is strongly monotone, by the classical maximum principle for elliptic equations. Indeed, this can be derived from the maximum principle for parabolic equation by observing that, by Duhamel's principle, we have

$$Tf = \int_0^\infty \Phi_t[f]\, dt,$$

where $\Phi_t : C(\overline{\Omega}) \to C(\overline{\Omega})$ is the semigroup operator so that $\Phi_t[u_0] = u(\cdot, t)$, with $u(x,t)$ being the unique solution to the initial boundary value problem

$$\begin{cases} u_t + \mathcal{L}u = 0 & \text{in } \Omega_T, \\ Bu = 0 & \text{on } S\Omega_T, \\ u(x,0) = u_0(x) & \text{in } \Omega. \end{cases}$$

Since Φ_t is strongly monotone for each $t > 0$ (thanks to the Strong Maximum Principle for linear parabolic equations), so is T.

Finally, we invoke the strong version of the Krein-Rutman Theorem (Theorem 1.3.4) to deduce that T has a principal eigenvalue Λ_1, in the sense that (i) $\Lambda_1 \in \mathbb{R}$ is a simple eigenvalue of T; (ii) $\Lambda_1 > \sup|\Lambda|$, where the supremum is taken over all other eigenvalues of T; (iii) the eigenfunction ϕ_1 corresponding to Λ_1 can be chosen so that $\phi_1 > 0$ in $\overline{\Omega}$; (iv) if Λ is an eigenvalue of T with a nonnegative eigenfunction, then $\Lambda = \Lambda_1$. By the definition of T, we deduce that the problem (1.23) has the principal eigenvalue $\mu_1 = \frac{1}{\Lambda_1}$ with the desired properties, except for assertion 2.

We prove assertion 2 by adapting an argument from [11, Chapter 6]. Given any other eigenvalue $\mu \in \mathbb{C}$ of (1.23) such that $\operatorname{Re}\mu \le \mu_1$, we will show that $\mu = \mu_1$.

Indeed, let $\mathcal{L}u = \mu u$ for some $u \not\equiv 0$, set

$$v = \frac{u}{\phi_1},$$

then direct computation yields

$$\mu v = \frac{1}{\phi_1}\mathcal{L}(v\phi_1) = \mathcal{L}v + cv - \frac{2}{\phi_1}a^{ij}D_j\phi_1 D_i v + \frac{v}{\phi_1}\mathcal{L}\phi_1. \tag{1.26}$$

Writing $\tilde{b}^i = b^i + \frac{1}{\phi_1}a^{ij}D_j\phi_1$ and

$$Kv := -a^{ij}D_{ij}v - \tilde{b}^i D_i v,$$

we deduce from (1.26) that

$$Kv = (\mu - \mu_1)v \quad \text{ in } \Omega. \tag{1.27}$$

Taking complex conjugate, we also derive

$$K\bar{v} = (\bar{\mu} - \mu_1)\bar{v} \quad \text{ in } \Omega. \tag{1.28}$$

Next, we compute

$$K(|v|^2) = K(v\bar{v}) = \bar{v}Kv + vK\bar{v} - 2a^{ij}D_i v D_j v \le \bar{v}Kv + vK\bar{v} - 2\lambda_0|Dv|^2, \tag{1.29}$$

where we used

$$a^{ij}\xi_i\bar{\xi}_j = a^{ij}[\operatorname{Re}\xi_i \operatorname{Re}\xi_j + \operatorname{Im}\xi_i \operatorname{Im}\xi_j] \ge \lambda_0|\xi|^2.$$

Substituting (1.27) and (1.28) into (1.29), we obtain

$$K(|v|^2) \le 2(\operatorname{Re}\mu - \mu_1)|v|^2 - 2\lambda_0|Dv|^2 \quad \text{ in } \Omega.$$

If $\operatorname{Re}\mu \le \mu_1$, we claim that v is constant, i.e. $u \in \operatorname{span}\{\phi_1\}$. Indeed, let $w = (\sup_\Omega |v|^2) - |v|^2$, then w satisfies

$$\begin{cases} Kw \ge 2\lambda_0|Dv|^2 \ge 0 \quad \text{and} \quad w \ge, \not\equiv 0 & \text{in } \Omega, \\ p^j D_j w = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.30}$$

By construction, $\inf_\Omega w = 0$. By the touching lemma (see Lemma 1.3.5), it follows that $w \equiv 0$ in Ω. Substituting this back into (1.30), we have that $|Dv| \equiv 0$ in Ω. This implies that $u \in \operatorname{span}\{\phi_1\}$. Hence, $\mu = \mu_1$ and the proof is complete. □

The following lemma gives a characterization of the maximum principle by the existence of a positive strict supersolution. The reader may skip the proof on first reading.

Lemma 1.3.9 *Suppose that the assumptions of Theorem* 1.3.6 *hold. Let* Ω *be a bounded domain with smooth boundary* $\partial\Omega$*, and let* $\Omega' \subset \Omega$ *be a proper subdomain. Suppose there exists* $w \in C(\overline{\Omega}')$ *such that* $w > 0$ *in* $\overline{\Omega}'$ *and*

$$\begin{cases} \mathcal{L}w \geq 0 & \text{in } \Omega', \\ \mathcal{B}w \geq 0 & \text{in } (\partial\Omega') \cap (\partial\Omega) \end{cases} \tag{1.31}$$

in the generalized sense. Then for any $u \in C(\overline{\Omega})$ *such that*

$$\begin{cases} \mathcal{L}u \leq 0 & \text{in } \Omega', \\ \mathcal{B}u \leq 0 & \text{in } (\partial\Omega') \cap (\partial\Omega), \\ u \leq 0 & \text{on } (\partial\Omega') \cap \Omega \end{cases} \tag{1.32}$$

in the generalized sense, we have

$$u \equiv 0 \quad \text{in } \Omega', \quad \text{or} \quad u < 0 \quad \text{in } \Omega' \cup [(\partial\Omega') \cap (\partial\Omega)].$$

Proof Denote

$$\partial\Omega' = \Gamma_1 \cup \Gamma_2, \quad \text{where } \Gamma_1 = (\partial\Omega') \cap (\partial\Omega), \quad \Gamma_2 = (\partial\Omega') \cap \Omega.$$

Since Ω' is a proper subset of Ω, and both are connected, Γ_2 is a nonempty set on which $w > 0 \geq u$. Hence $u \not\equiv kw$ for all $k > 0$. By the strong maximum principle, it suffices to show that $u \leq 0$ in Ω'. Suppose not, then there exists a constant $k > 0$ such that $v = u - kw$ satisfies

$$v \leq 0 \quad \text{in } \Omega', \quad \text{and } v(x_0) = 0 \quad \text{for some } x_0 \in \overline{\Omega}',$$

and also satisfies (1.32) in the generalized sense. On the one hand, the strong maximum principle implies $x_0 \notin \Omega'$. On the other hand, $w > 0$ on $\overline{\Gamma}_2$, so that $v < 0$ on $\overline{\Gamma}_2$ and hence $x_0 \notin \overline{\Gamma}_2$. Hence, $v < 0$ in Ω' and $v(x_0) = 0$ for some $x_0 \in \partial\Omega' \setminus \overline{\Gamma}_2$. Therefore, x_0 lies on the interior of Γ_1, which is the smooth part of $\partial\Omega'$. The Hopf Boundary Lemma implies

$$\mathcal{B}v = p^i D_i v + p^0 v = p^i D_i v > 0 \quad \text{at } x_0.$$

This is a contradiction, since $\mathcal{B}v = \mathcal{B}u - k\mathcal{B}w \leq 0$. □

Corollary 1.3.10 *Suppose that the assumptions of Theorem* 1.3.6 *hold. Let* Ω *be a bounded domain with smooth boundary* $\partial\Omega$*. Suppose there exists* $w \in C(\overline{\Omega})$ *such that* $w > 0$ *in* $\overline{\Omega}$ *and*

$$\begin{cases} \mathcal{L}w \geq 0 & \textit{in } \Omega, \\ \mathcal{B}w \geq 0 & \textit{in } \partial\Omega \end{cases} \tag{1.33}$$

in the generalized sense. Then for any $u \in C(\overline{\Omega})$ *such that*

$$\begin{cases} \mathcal{L}u \leq 0 & \textit{in } \Omega, \\ \mathcal{B}u \leq 0 & \textit{in } \partial\Omega \end{cases} \tag{1.34}$$

in the generalized sense, one of the followings holds.

- $u < 0$ *in* $\overline{\Omega}$*;*
- $u \equiv 0$ *in* $\overline{\Omega}$*;*
- $u \equiv kw$ *in* Ω*, and the equalities hold in* (1.33) *in the classical sense.*

Remark 1.3.11 In case w is a strict positive supersolution, then it must hold that $u \leq 0$, i.e. the maximum principle holds. In case there are no strict positive supersolutions, then it must hold that $\mu_1 \leq 0$, where μ_1 is the principal eigenvalue of (1.23) (otherwise the corresponding eigenfunction $\phi_1 > 0$ is a strict positive supersolution). In such a case the maximum principle fails as

$$\mathcal{L}\phi_1 \leq 0 \quad \text{in } \Omega, \quad \mathcal{B}\phi_1 = 0$$

but $\phi_1 > 0$ in $\overline{\Omega}$.

Proof If $u \leq 0$ in Ω, then the strong maximum principle asserts that either $u \equiv 0$ or $u < 0$ in $\overline{\Omega}$. If $u > 0$ somewhere in Ω, repeat the proof of Lemma 1.3.9 with $\Omega' = \Omega$, then there exists $k > 0$ such that $v = u - kw$ satisfies either $v \equiv 0$ or $v < 0$ in $\overline{\Omega}$. By the choice of k, we must have $v \equiv 0$. This implies that $u \in \text{span}\{w\}$. □

One can estimate the principal eigenvalue by constructing appropriate super/subsolutions, as the following two lemmas illustrate. For simplicity, we assume in addition that $a^{ij}, b^i, c \in C^\alpha(\overline{\Omega})$ and $p^i \in C^{1+\alpha}(\overline{\Omega})$, so that the principal eigenfunction $\phi \in C^{2+\alpha}(\overline{\Omega})$ thanks to the Schauder theory (see [28, Chapter 2]).

Lemma 1.3.12 *Suppose that the assumptions of Theorem* 1.3.6 *hold and let* μ_1 *be the principal eigenvalue of* (1.23)*. Suppose, in addition, that there exist a function* $w \in C(\overline{\Omega})$ *and a constant* $\underline{\mu}$ *such that* $w > 0$ *in* $\overline{\Omega}$ *is a supersolution in the sense that*

$$\mathcal{L}w \geq \underline{\mu}w \quad \textit{in } \Omega, \quad \textit{and} \quad \mathcal{B}w \geq 0 \quad \textit{on } \partial\Omega \tag{1.35}$$

in the generalized sense. Then $\mu_1 \geq \underline{\mu}$ *and the equality holds if and only if* w *is the corresponding eigenfunction, i.e. the equalities hold in* (1.35)*.*

Proof Let μ_1 and $\phi > 0$ be the principal eigenvalue and positive eigenfunction of (1.23). Define $u = \phi - kw$, where $k > 0$ is the least real number such that $\phi_1 - kw \leq 0$.

Suppose that $\mu_1 \leq \underline{\mu}$, it remains to show that $\mu_1 = \underline{\mu}$ and w satisfies (1.23) in the classical sense. Indeed, Corollary 1.3.10 implies that $u = \phi - kw \in \text{span}\{\phi\}$. By the choice of k, we must have $u = 0$. Hence $w \in \text{span}\{\phi\}$. □

Lemma 1.3.13 *Suppose that the assumptions of Theorem* 1.3.6 *hold and let* μ_1 *be the principal eigenvalue of* (1.23)*. Suppose, in addition, that there exist a nonnegative and nontrivial function* $w \in C(\overline{\Omega})$ *and a constant* $\overline{\mu}$ *such that* w *is a subsolution of* (1.23)*, i.e.*

$$\mathcal{L}w \leq \overline{\mu}w \quad \text{in } \Omega, \quad \text{and} \quad \mathcal{B}w \leq 0 \quad \text{on } \partial\Omega, \tag{1.36}$$

in the generalized sense. Then $\mu_1 \leq \overline{\mu}$*, and the equality holds if and only if* w *is the corresponding eigenfunction, i.e. the equality holds in* (1.36)*.*

Proof Let μ_1 and $\phi > 0$ be the principal eigenvalue and positive eigenfunction of (1.23). Define $u = w - k\phi$, where $k > 0$ is the least real number such that $\phi_1 - kw \leq 0$.

Suppose that $\mu_1 \geq \overline{\mu}$, it remains to show that $\mu_1 = \overline{\mu}$ and w satisfies (1.23) in the classical sense. Indeed, Corollary 1.3.10 implies that $u = w - k\phi \in \text{span}\{\phi\}$. By the choice of k, we must have $u = 0$. Hence $w \in \text{span}\{\phi\}$. □

We end this chapter with a classical result: the monotonicity dependence of eigenvalue on the growth rate $c(\cdot)$ and the diffusion rate d. Consider the following elliptic eigenvalue problem.

$$\begin{cases} -d\Delta\phi - c\phi = \mu\phi & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla\phi + p^0\phi = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.37}$$

where $d > 0$ and $c \in C^{\alpha}(\overline{\Omega})$ and $0 \leq p^0 \in C^{1+\alpha}(\overline{\Omega})$ for some $\alpha \in (0, 1)$. It follows that the principal eigenfunction $\phi_1 \in C^{2+\alpha}(\overline{\Omega})$ satisfies (1.37) in the classical sense.

Lemma 1.3.14 *Let* $\mu_1(c)$ *be the principal eigenvalue of* (1.37)*, and let* $\mu_1(c')$ *be the principal eigenvalue of* (1.37) *with* c *replaced by* c'*. If* $c \leq c'$ *in* Ω*, then* $\mu_1(c) \geq \mu_1(c')$*.*

Proof This follows from Lemma 1.3.12. □

We prove the well-known results concerning the monotonicity of eigenvalue with respect to the diffusion rate. Note that this monotonicity fails in general when the drift is nonzero.

Proposition 1.3.15 *Let* μ_1 *be the principal eigenvalue of* (1.37)*. Then* $\mu_1 = \mu_1(d)$ *is smooth and nondecreasing in the diffusion rate* $d > 0$*. If, in addition,* c *is nonconstant or* $p^0 \not\equiv 0$*, then* $\frac{\partial}{\partial d}\mu_1 > 0$ *for* $d > 0$*.*

Proof By scaling, we may assume without loss of generality that $|\Omega| = 1$. Fix a positive eigenfunction ϕ_1 corresponding to μ_1 and normalize it by $\int_\Omega |\phi_1|^2 = 1$.

Assuming smoothness, we denote $'$ to be the partial derivative with respect to diffusion rate d, and differentiate (1.37) to obtain

$$\begin{cases} d\Delta\phi_1' + c\phi_1' + \mu_1\phi_1' + \Delta\phi_1 = -\mu_1'\phi_1 & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla\phi_1' + p^0\phi_1' = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.38}$$

We claim that

$$\int_\Omega \phi_1 \Delta \phi_1' = \int_\Omega \phi_1' \Delta \phi_1. \tag{1.39}$$

Indeed,

$$\begin{aligned}
\int_\Omega \phi_1 \Delta \phi_1' &= -\int_\Omega \nabla \phi_1 \cdot \nabla \phi_1' + \int_{\partial\Omega} \phi_1 (\mathbf{n} \cdot \nabla \phi_1') \\
&= \int_\Omega \phi_1' \Delta \phi_1 + \int_{\partial\Omega} \left[\phi_1 (\mathbf{n} \cdot \nabla \phi_1') - \phi_1' (\mathbf{n} \cdot \nabla \phi_1) \right] \\
&= \int_\Omega \phi_1' \Delta \phi_1 + \int_{\partial\Omega} \left[\phi_1 (-p^0 \phi_1') - \phi_1' (-p^0 \phi_1) \right] = \int_\Omega \phi_1' \Delta \phi_1,
\end{aligned}$$

where we used the homogeneous Robin boundary condition.

Multiply (1.38) by ϕ_1 and using (1.39), we obtain

$$\begin{aligned}
-\mu_1' \int_\Omega |\phi_1|^2 &= \int_\Omega \phi_1 (d\Delta \phi_1' + c\phi_1' + \mu_1 \phi_1') + \int_\Omega \phi_1 \Delta \phi_1 \\
&= \int_\Omega \phi_1 (d\Delta \phi_1' + c\phi_1' + \mu_1 \phi_1') - \int_\Omega |\nabla \phi_1|^2 + \int_{\partial\Omega} \phi_1 (\mathbf{n} \cdot \nabla \phi_1) \\
&= \int_\Omega \phi_1' (d\Delta \phi_1 + c\phi_1 + \mu_1 \phi_1) - \int_\Omega |\nabla \phi_1|^2 - \int_{\partial\Omega} p^0 |\phi_1|^2 \\
&= -\int_\Omega |\nabla \phi_1|^2 - \int_{\partial\Omega} p^0 |\phi_1|^2 \le 0.
\end{aligned}$$

This proves that $\mu_1' \ge 0$ for all $d > 0$, i.e. $d \mapsto \mu_1$ is nondecreasing. If, in addition, either c is nonconstant, or $p^0 \not\equiv 0$, then ϕ_1 is nonconstant, so that $\mu_1' > 0$ for all $d > 0$.

It remains to prove the smooth dependence of (μ_1, ϕ_1) on d. We use an idea due to A. Lazer (see [4, Lemma 1.2]) via applying the Implicit Function Theorem by regarding (μ_1, ϕ_1) as the unique solution of the mapping $\mathcal{F} : \mathbb{R} \times Y \times (0, \infty) \to C^\alpha(\overline{\Omega})$, given by

$$\mathcal{F} \begin{pmatrix} \mu \\ \phi \\ d \end{pmatrix} = \begin{pmatrix} d\Delta \phi + c\phi + \mu\phi \\ \frac{1}{2} \int_\Omega |\phi|^2 \, dx \end{pmatrix},$$

where

$$Y = \{\phi \in C^{2+\alpha}(\overline{\Omega}) : \ \mathbf{n} \cdot \nabla \phi + p^0 \phi = 0 \ \text{ on } \partial\Omega\}.$$

For each $d > 0$, Theorem 1.3.6 asserts the existence of $(\mu_1, \phi_1) = (\mu_1(d), \phi_1(x; d))$ such that

$$\mathcal{F}(\mu_1(d), \phi_1(\cdot; d), d) = 0.$$

To prove the smooth dependence of (μ_1, ϕ_1) on d, it suffices to show that for each fixed $d > 0$, the linear mapping

$$D_{(\mu,\phi)} \mathcal{F}(\mu_1(d), \phi_1(\cdot; d), d) : \mathbb{R} \times Y \to C^\alpha(\overline{\Omega}) \times \mathbb{R}$$

is invertible. To this end, given $(f, \bar{g}) \in C^{\alpha}(\overline{\Omega}) \times \mathbb{R}$, we need to prove the existence and uniqueness of $(\bar{h}, w) \in \mathbb{R} \times Y$ such that

$$\begin{cases} d\Delta w + cw + \mu_1 w + \bar{h}\phi_1 = f & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla w + p^0 w = 0 & \text{on } \partial\Omega, \\ \int_\Omega \phi_1 w = \bar{g}. \end{cases} \tag{1.40}$$

First we show existence. To this end, we choose $\bar{h} = \int_\Omega f\phi_1$, so that

$$\int_\Omega (f - \bar{h}\phi_1)\phi_1 = \left(\int_\Omega f\phi_1\right) - \bar{h} = 0. \tag{1.41}$$

Next, we define $\hat{w} \in W^{1,2}(\Omega)$ to be the unique weak solution to

$$\begin{cases} d\Delta \hat{w} + c\hat{w} + \mu_1 \hat{w} = f - \bar{h}\phi_1 & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla \hat{w} + p^0 \hat{w} = 0 & \text{on } \partial\Omega, \\ \int_\Omega \hat{w}\phi_1 = 0. \end{cases}$$

Such a choice of $\hat{w}$ is well-defined in view of the Fredholm alternative [15, Theorem 5.11], thanks to (1.41) and the fact that μ_1 is a simple eigenvalue with eigenfunction ϕ_1. Furthermore, $\hat{w} \in C^{2+\alpha}(\overline{\Omega})$ according to elliptic regularity theory [28, Theorem 4.40]. Finally, define $w = \hat{w} + \bar{g}\phi_1$, and observe that $(\bar{h}, w)$ satisfies (1.40). This proves existence.

To show uniqueness, we set $(f, \bar{g}) = (0, 0)$ in (1.40) and proceed to show $(\bar{h}, w) = (0, 0)$. Indeed, multiplying the first equation of (1.40) by ϕ_1, and integrating, we have

$$-\bar{h}\int_\Omega |\phi_1|^2 = \int_\Omega \phi_1(d\Delta w + cw + \mu_1 w) = \int_\Omega w(d\Delta\phi_1 + c\phi_1 + \mu_1\phi_1) = 0,$$

where we used the Robin boundary condition of w and ϕ_1, as we integrate by parts twice for the second equality. Thus $\bar{h} = 0$ and (1.40) becomes

$$\begin{cases} d\Delta w + cw + \mu_1 w = 0 & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla w + p^0 w = 0 & \text{on } \partial\Omega, \\ \int_\Omega w\phi_1 = 0. \end{cases}$$

Since μ_1 is a simple eigenvalue, it follows that $w = k\phi_1$ for some $k \in \mathbb{R}$, so that the integral condition becomes $\int_\Omega k|\phi_1|^2 = 0$. Since $\phi_1 \neq 0$, we have $k = 0$. i.e. $w = 0$. This completes the proof for the smooth dependence of (μ_1, ϕ_1) on d. □

Proposition 1.3.16 *Let μ_1 be the principal eigenvalue of* (1.37)*, where $p^0 \geq 0$. Then*

$$\mu_1 \geq -\max_{\overline{\Omega}} c. \tag{1.42}$$

Furthermore, we have

$$\lim_{d\to 0+} \mu_1 = -\max_{\overline{\Omega}} c. \tag{1.43}$$

Proof First, we integrate the first equation of

$$\begin{cases} -d\Delta\phi_1 - c\phi_1 = \mu_1\phi_1 & \text{in } \Omega, \\ \mathbf{n}\cdot\nabla\phi_1 + p^0\phi_1 = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.44}$$

over Ω to obtain

$$-\int_\Omega c\phi_1 \le \mu_1 \int_\Omega \phi_1. \tag{1.45}$$

In particular, (1.42) holds. Note that (1.42) can also be established by applying Lemma 1.3.12, which is the comparison principle for the principal eigenvalues; See Problem **??**.

Next, we show (1.43). For this purpose, fix $x_0 \in \Omega$ and $0 < r < \text{dist}(x_0, \partial\Omega)$, and let $\tilde{\lambda}$ and $\tilde{\phi}$ be the principal eigenvalue and the positive eigenfunction of

$$-\Delta\tilde{\phi} = \tilde{\lambda}\tilde{\phi} \quad \text{in } B_r(x_0), \quad \tilde{\phi} = 0 \quad \text{on } \partial B_r(x_0).$$

Since $\phi_1 > 0$ in $\overline{B_r(x_0)}$ and $\tilde{\phi} = 0$ on $\partial B_r(x_0)$, up to multiplication of $\tilde{\phi}$ by a positive constant, we may assume that

$$\phi_1 \ge \tilde{\phi} \quad \text{in } B_r(x_0) \quad \text{and} \quad \phi_1(x_0') = \tilde{\phi}(x_0') > 0 \quad \text{for some } x_0' \in B_r(x_0).$$

Therefore, $\Delta\phi_1(x_0') \ge \Delta\tilde{\phi}(x_0')$. Evaluating (1.44) at the point x_0', we have

$$d\tilde{\lambda}\tilde{\phi}(x_0') = -d\Delta\tilde{\phi}(x_0') \ge (c(x_0') + \mu_1)\tilde{\phi}(x_0').$$

Divide by $\tilde{\phi}(x_0') > 0$, we have

$$\mu_1 \le -c(x_0') + d\tilde{\lambda} \le -\inf_{B_r(x_0)} c + d\tilde{\lambda}.$$

Taking lim sup as $d \to 0$, and then sending $r \to 0$, we have

$$\limsup_{d\to 0} \mu_1 \le -c(x_0) \quad \text{for each } x_0 \in \Omega.$$

Since $x_0 \in \Omega$ is arbitrary, we obtain

$$\limsup_{d\to 0} \mu_1 \le -\max_{\overline{\Omega}} c. \tag{1.46}$$

Combining with the first inequality in (1.42), we obtain (1.43). □

Next, we prove an elementary result of semi-classical analysis.

Proposition 1.3.17 *Let μ_1 be the principal eigenvalue of* (1.37)*, where $p^0 \ge 0$. Let ϕ_1 be a positive eigenfunction associated with μ_1, which is normalized by* $\sup_\Omega \phi_1 = 1$*, then as $d \to 0$,*

$$\phi_1 \to 0 \quad \textit{locally uniformly in } \Omega_0,$$

where $\Omega_0 = \{x \in \overline{\Omega} : \ c(x) < \sup_\Omega c\}$.

Remark 1.3.18 In particular, the principal eigenfunction ϕ_1 (normalized by $\sup_\Omega \phi_1 = 1$) concentrates on the set where c attains its global maximum value.

Proof Set $d = \epsilon^2$, then ϕ_1 and μ_1 satisfies

$$-\epsilon^2 \Delta \phi_1 = (c(x) + \mu_1)\phi_1 \quad \text{ in } \Omega.$$

Next, we introduce the WKB-ansatz (see [13])

$$w_\epsilon = -\epsilon \log \phi_1.$$

Then $w_\epsilon \geq 0$, and satisfies

$$\begin{cases} -\epsilon \Delta w_\epsilon + |\partial_x w_\epsilon|^2 + c(x) + \mu_1 = 0 & \text{in } \Omega, \\ \mathbf{n} \cdot \nabla w_\epsilon \geq 0 & \text{on } \partial\Omega, \\ \inf_\Omega w_\epsilon = 0. \end{cases} \tag{1.47}$$

Define the semi-relaxed limit

$$w_*(x) := \liminf_{\substack{\epsilon \to 0 \\ x' \to x}} w_\epsilon(x').$$

Since w_ϵ are nonnegative, we deduce that w_* is a well-defined nonnegative function in $\overline{\Omega}$, if we allow the possibility that $w_* = +\infty$ at certain points. The following claim uses the concept of viscosity solution implicitly.

Claim $w_* : \overline{\Omega} \to [0, \infty]$ is lower semicontinuous in $\overline{\Omega}$, $\inf_{\overline{\Omega}} w_* = 0$, and satisfies

$$w_*(x) > 0 \quad \text{ in } \Omega_0 = \{x \in \overline{\Omega} : \ c(x) < \sup_\Omega c\}. \tag{1.48}$$

The lower semicontinuity follows by construction. Next, choose a point $x_\epsilon \in \overline{\Omega}$ such that $w_\epsilon(x_\epsilon) = 0$, we can then pass to a sequence such that $x_\epsilon \to x_1$ for some $x_1 \in \bar{\Omega}$. It then follows that $w_*(x_1) \leq 0$. Since w_* is nonnegative, this proves that $\inf_{\overline{\Omega}} w_* = 0$.

Next, suppose to the contrary that there exists $x_0 \in \overline{\Omega}$ such that $c(x_0) < \sup_\Omega c$ and $w_*(x_0) = 0$. Then $w_* + |x - x_0|^2$ has a strict local minimum at x_0. By construction, there exists $y_\epsilon \in \Omega$ such that $y_\epsilon \to x_0$ such that $w_\epsilon + |x - x_0|^2$ attains a local minimum at y_ϵ. We first consider the case x_0 is an interior point, wherein y_ϵ are interior points as well. Then

$$\nabla(w_\epsilon + |x - x_0|^2) = 0 \quad \text{ and } \quad \Delta(w_\epsilon + |x - x_0|^2) \geq 0 \quad \text{ at } y_\epsilon,$$

so that

$$c(y_\epsilon) + \mu_1 = -|\partial_x w_\epsilon(y_\epsilon)|^2 + \epsilon\Delta w_\epsilon(y_\epsilon) \geq -|2(y_\epsilon - x_0)|^2 - 2N\epsilon,$$

where N is the dimension of Ω. Letting $\epsilon \to 0$, then $\mu_1 \to -\sup_\Omega c$ (in view of Proposition 1.3.16) and we obtain

$$c(x_0) - \sup_\Omega c \geq 0.$$

This contradicts $c(x_0) < \sup_\Omega c$. Hence, (1.48) holds in case $x_0 \in \Omega$. In case $x_0 \in \partial\Omega$, we consider instead the local minimum point y_ϵ of $w_\epsilon(x) + |x - x_0 + \epsilon n_0|^2$, where n_0 is the outer unit normal vector at x_0. Using the boundary condition of w_ϵ, there exists $r > 0$ such that for all ϵ small,

$$\mathbf{n} \cdot \nabla(w_\epsilon(x) + |x - x_0 + \epsilon n_0|^2) > 0 \quad \text{on } \partial\Omega \cap B_r(x_0).$$

It follows that the local minimum y_ϵ is attained in the interior, and one can argue similarly as in the case $x_0 \in \Omega$. Hence, (1.48) holds and the claim is proved.

Next, define

$$\Omega_0^\delta = \{x \in \Omega : c(x) < \sup_\Omega c - \delta\},$$

where $\delta > 0$ is the same as in the beginning of the proof. By the lower semicontinuity of w_*, the following positive number is well-defined.

$$\eta := \frac{1}{2} \inf_{\Omega_0^\delta} w_* > 0.$$

Hence, there exists $\delta' \in (0, \delta)$ such that for $\epsilon \in (0, \delta')$, we have $\inf_{\Omega_0^\delta} w_\epsilon \geq \eta$. Hence, we deduce that

$$\sup_{\{x \in \Omega:\ c(x) < \sup c - \delta\}} \phi_1 \leq e^{-\frac{\eta}{\epsilon}} \quad \text{for all } 0 < \epsilon \ll 1.$$

Since δ is arbitrary, this proves that $\phi_1 \to 0$ in $C_{loc}\left(\{x \in \overline{\Omega} : c(x) < \max_{\overline{\Omega}} c\}\right)$. □

Proposition 1.3.19 *Let μ_1 and ϕ_1 be the principal eigenvalue and positive eigenfunction of* (1.37). *If $p^0 \equiv 0$, then*

$$-\max_{\overline{\Omega}} c \leq \mu_1 \leq -\frac{1}{|\Omega|} \int_\Omega c. \tag{1.49}$$

Furthermore, we have

$$\lim_{d \to 0+} \mu_1 = -\max_{\overline{\Omega}} c, \tag{1.50}$$

$$\lim_{d \to \infty} \mu_1 = -\frac{1}{|\Omega|} \int_\Omega c, \tag{1.51}$$

and that, as $d \to \infty$, $\phi_1 \to 1$ weakly in $W^{2,p}(\Omega)$ and strongly in $C^1(\overline{\Omega})$.

Proof First, we integrate

$$\begin{cases} -d\Delta\phi_1 - c\phi_1 = \mu_1\phi_1 & \text{in } \Omega, \\ \mathbf{n}\cdot\nabla\phi_1 = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.52}$$

over Ω to obtain

$$-\int_\Omega c\phi_1 = \mu_1 \int_\Omega \phi_1. \tag{1.53}$$

In particular,

$$-\max_{\overline{\Omega}} c \le \mu_1 \le -\min_{\overline{\Omega}} c. \tag{1.54}$$

Next, divide (1.52) by ϕ_1 (note that $\phi_1 > 0$ in $\overline{\Omega}$) and integrate by parts,

$$|\Omega|\mu_1 = -d\int_\Omega \frac{|\nabla\phi_1|^2}{|\phi_1|^2} - \int_\Omega c \le -\int_\Omega c. \tag{1.55}$$

Combining (1.54) and (1.55), we derive (1.49). We also note that (1.50) is proved in Proposition 1.3.19.

To show (1.51), we normalize the principal eigenfunction ϕ_1 by $\sup_\Omega \phi_1 = 1$. Dividing (1.52) by d, we have

$$-\Delta\phi_1 = \frac{1}{d}(c + \mu_1)\phi_1. \tag{1.56}$$

By using elliptic L^p estimates, we can pass to a subsequence to get $\phi_1 \to \phi_\infty$ weakly in $W^{2,p}(\Omega)$ and strongly in $C^1(\overline{\Omega})$, where ϕ_∞ is a strong solution of

$$-\Delta\phi_\infty = 0 \quad \text{in } \Omega, \quad \text{and} \quad \mathbf{n}\cdot\nabla\phi_\infty = 0 \quad \text{on } \partial\Omega.$$

i.e. ϕ_∞ is a constant. Since $\phi_1 \to \phi_\infty$ uniformly and $\sup_\Omega \phi = 1$, it follows that $\phi_\infty \equiv 1$. In conclusion, $\phi_1 \to 1$ in $C(\overline{\Omega})$ as $d \to \infty$. Letting $d \to \infty$ in (1.53), we obtain the desired result. □

To close the chapter, we consider the one-dimensional domain $\Omega = (0, L)$, and state a result concerning the dependence on drift.

Lemma 1.3.20 *Let μ_1 be the principal eigenvalue of*

$$d\psi'' + \alpha\psi' + c\psi + \mu_1\psi = 0 \quad \textit{in } (0, L), \quad \textit{and} \quad \psi'(0) = \psi'(L) = 0.$$

If $c(x)$ is nonconstant and nonincreasing, then $\frac{\partial}{\partial\alpha}\mu_1 > 0$ for $\alpha \in \mathbb{R}$.

Proof Refer to [19, Lemma 5.2] or Lemma 8.3.7 in Chapter 8. □

1.4 Further reading

In this text, we assume the knowledge of the regularity theory for elliptic and parabolic equations. A nice overview of the basic theory can be found in [20] with complete references to find specific proofs. The classical reference for regularity of elliptic equations is [15], and a more recent account of the oblique derivative problem can be found in [28]; see also [8]. For regularity theory of parabolic equations, see [14, 26]. An alternative to construct solutions to parabolic equations is via semigroup theory, which is based on the variation of constants formulation. This latter approach relies only on elliptic regularity estimates; see [16, 29].

An influential paper on monotone methods for elliptic and parabolic equations is [32], which contains in particular Corollary 1.2.4.

The classical reference for maximum principles is [31], in which the maximum principle in more general, noncylindrical domains, are treated for both elliptic and parabolic equations.

We applied the Krein-Rutman Theorem 1.3.3, for compact linear operator preserving a cone, to derive the principal eigenvalue for elliptic problems; see Appendix B for the proof of various versions of the Krein-Rutman Theorem. Our estimate for the spectral gap (assertion 2 of Theorem 1.3.6) is taken from [11, Chapter 6]. The characterization of maximum principle by principal eigenvalue is a classical result; see, e.g. [3] for the Dirichlet case in general bounded domains. A more practical “rule of thump" can be stated as follows: for a given linear elliptic boundary value problem, the validity of the maximum principle is equivalent to the existence of a strict positive supersolution; see Corollary 1.3.10.

We also mention here the recent work by Chang et al. [5], where the authors prove in an elementary way the strong version of the Krein-Rutman Theorem for semi-strongly positive operators (including strongly positive operators), and study the relationship between the semi-strong positivity and the ideal-irreducibility in a Banach lattice, as well as the upper and lower spectral radii for reducible linear positive operators. For reducible operators, they prove that the lower spectral radius always serves as the least upper bound of the set of eigenvalues pertaining to positive eigenvectors, and the upper spectral radius the greatest lower bound of the set. Moreover, they apply these abstract results on some PDE examples.

The smooth dependence of principal eigenvalue, proved in Proposition 1.3.15 is due to an idea of Lazer, which appeared in [4]. See also Chapter 4 for the recent generalization to principal Floquet bundle.

In the absence of advection, we proved the monotonicity of the principal eigenvalue in diffusion rate in Proposition 1.3.15. This is connected to the so-called *reduction principle* due to L. Altenberg [1] concerning the monotonicity of the spectral bound $s(\rho A + Q)$ in $\rho \in \mathbb{R}_+$, where A is a compact linear operator in a Banach space with positive resolvents, and Q is a multiplicative operator. This, in turn, is a generalization of a theorem due to Karlin [22] for the finite dimensional case; See also [6, 23].

In Proposition 1.3.17, we use the WKB-ansatz to show that the eigenfunction of (1.37), when normalized in L^1, tends to a measure μ supported on the set of global maximum points of the zero-th order coefficient. The determination of this measure is known as the semi-classical limits; See [17] and the references therein. In general, if the zero-th order coefficient is a Morse function, then the set of global maximum points is finite and the weight of μ at each of these point is determined by the associated Hessian.

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